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NUMERICAL ANALYSIS OF SALTWATER INTRUSION USING B-SPLINE BOUNDARY ELEMENTS

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SUMMARY

Continuity of the derivatives of the main variable is an important feature to obtain an accurate representation of moving boundaries with discrete numerical methods, since the value and direction of the velocity are normally used to relocate the nodal points in a time-marching scheme. A recently developed formulation of the boundary element method using cubic B-splines provides up to C^2 continuity between adjacent elements. This formulation is applied in this work to saltwater intrusion problems in confined, leaky and unconfined aquifers.

KEY WORDS Boundary element method Saltwater intrusion B-splines

INTRODUCTION

This paper discusses the numerical solution of sharp-interface saltwater intrusion models using integral equations. The problem is formulated as a moving boundary problem for which, at each time step, a Laplace equation is solved by using a Boundary Element Method (BEM) to provide velocity values at the interface between fresh and salt water. The position of the interface is then moved by applying a finite difference algorithm to each nodal point along the interface. The same idea applies to the phreatic surface for the case of unconfined aquifers.

Moving boundary algorithms generally demand an accurate computation of both direction and modulus of the velocity at points located on the moving line or surface. When using a discretization procedure with Lagrangian interpolation (even of higher order) oscillations may develop due to the appearance, during the motion, of points of geometric tangent discontinuity between adjacent elements at which the velocity is not unique.

The above problem has been resolved, in the present work, by adopting a recently developed BEM formulation based on uniform and non-uniform cubic B-splines,[1,2] capable of preserving continuity of the tangent and curvature (C^2 continuity) between elements. This formulation has already been successfully applied to groundwater seepage problems by Cabral and Wrobel.[3]

Saltwater intrusion into coastal aquifers is a subject of concern which has been studied since the last century. van der Leeden[4] published a selected bibliography of experimental and theoretical research on this subject while Reilly and Goodman[5] presented a historical perspective of quantitative analysis of the saltwater–freshwater relationship in groundwater systems. More

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recently, important advances have been made using semi-analytical and numerical techniques for better modelling of steady-state and transient problems.

Boundary elements have been applied to saltwater intrusion studies by Awater[6] using a formulation in the complex plane, with application to a lake reclamation problem. Liu *et al.*[7] developed a formulation using real variables and linear elements for the case of confined aquifers while Kemblowski[8,9] applied a similar formulation to saltwater upconing problems. Taigbenu *et al.*[10] presented a different approach involving a modified potential and areal modelling. Cabral and Wrobel[11] expanded the formulation of Liu *et al.*[7] by including drains and trenches for simulating water exploitation in confined coastal aquifers while Cabral and Cirilo[12] have used a similar approach to model a semiconfined aquifer. de Lange[13] applied a simplified BEM model to analyse the behaviour of the saltwater–freshwater interface in a three-dimensional problem. Fukui *et al.*[14] also applied a three-dimensional formulation with constant flat elements for a lens-shaped intrusion in an island. The present paper extends the range of application of the BEM to saltwater intrusion into unconfined aquifers, which involves two moving boundaries, and where the motion of the phreatic surface and interface are interdependent.

Cubic B-splines are used in this work both for geometric representation and as interpolation functions. The numerical results obtained with the present formulation are compared with analytical and other numerical solutions showing accurate results with coarse discretizations.

MATHEMATICAL MODELLING

In a coastal aquifer, the flow may be assumed everywhere to be orthogonal to the shore line, so that a two-dimensional vertical model may be used to analyse the saltwater intrusion problem.

In many practical cases the thickness of the transition zone between fresh water and salt water is relatively small compared to the aquifer dimensions. It is thus a common approximation to neglect the transition zone and assume a sharp interface separating the two regions. Bear[15] has given some examples of real aquifers in which experimental measurements have shown the salt concentration to vary sharply at a determined location, clearly establishing a region of low salt concentration (fresh water) and a region of high salt concentration (salt water).

A further approximation can be introduced into the mathematical model by assuming a constant potential value in the saltwater region and analysing only the freshwater region (the Ghyben–Herzberg approximation, see Reference 15). This approach, although simpler, may lead to unsatisfactory results[13,16] and is not adopted here.

Some analytical and numerical models have been developed using the Dupuit approximation.[17,18] This assumption generally gives reasonable results for steady flow analysis if the aquifer is shallow, but it can yield significant errors in transient problems with sudden changes of flow rate.[19]

For the flow under a semipervious layer, Mualem and Bear[20] have shown that two different situations frequently occur in practice: the first, sketched in Figure 1(a), is the case when the interface actually intersects the semipervious layer; the other case, shown in Figure 1(b), is when the exceeding fresh water may leave the aquifer through an outflow face located after the end of the semipervious layer. Analytical and approximate solutions for the shape of the interface in a semiconfined aquifer, under the Dupuit assumption, have been found by Sikkema and van Dam[21,22] for 13 different situations, according to the physical parameters of the problem.

For saltwater intrusion in an unconfined aquifer, Vappicha and Nagaraja[23] obtained an approximate analytical solution by using a quasi-steady assumption which directly relates the movement of the interface with the free surface position. This is an oversimplification which may produce significant errors for all but the simplest cases.[18]

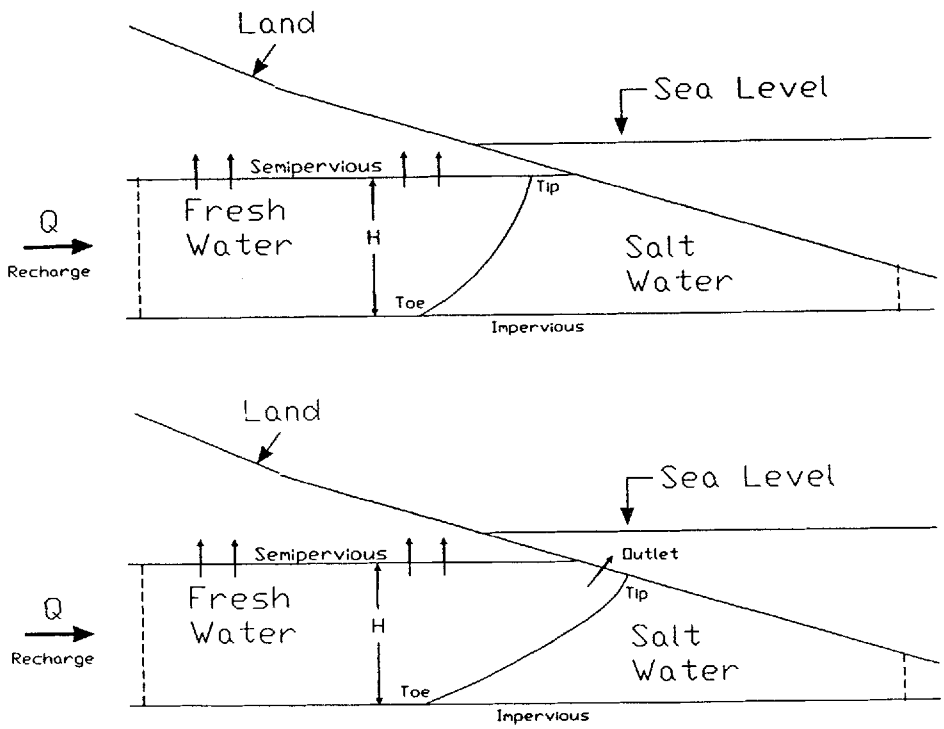

Figure 1. Saltwater intrusion in leaky aquifer: (a) interface intersecting semipervious layer; (b) outflow face

In unconfined aquifers, usually the phreatic surface intersects the land slope at some distance from the shore line (Figure 2, point B), forming a seepage face. Although in some problems this is not important, it should be taken into account for a more realistic modelling.[24]

In the present paper, a sharp-interface model in a two-dimensional vertical plane is adopted, and the freshwater and saltwater regions considered by employing a standard BEM subregions technique.[25]

MATHEMATICAL FORMULATION

A combination of Darcy's law and the mass balance equation for a two-dimensional flow through a homogeneous isotropic porous medium gives, for a vertical model,

$$K\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=S\,\frac{\partial u}{\partial t}, \tag{1}$$

where K is the hydraulic conductivity, S the specific storage and the potential u is the piezometric head

$$u=y+\frac{p}{\gamma},$$

in which p is pressure, γ is specific weight and y is elevation.

The specific storage S can generally be regarded as zero when the compressibility of the porous matrix is negligible.[26] In these situations, transient flows can be treated as a sequence of

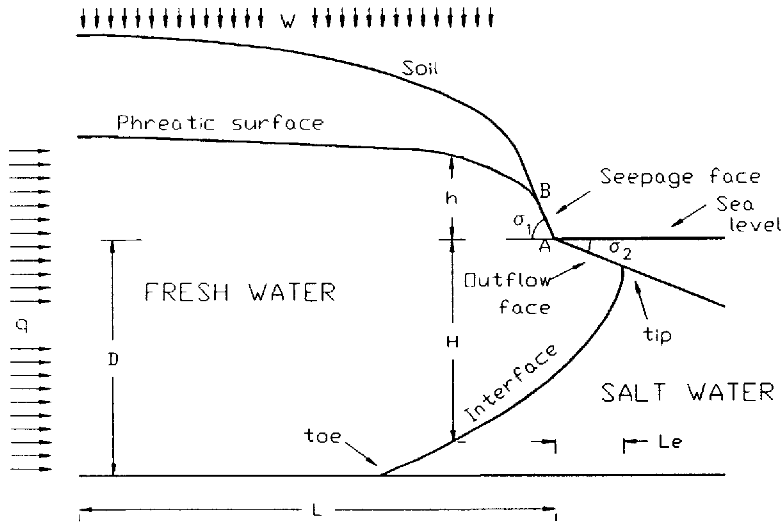

Figure 2. Saltwater intrusion in unconfined aquifer with accretion

successive steady states in a step-by-step procedure. So, the problem is modelled using Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{2}$$

In the present paper, at each time step, this equation is applied to each region independently, and compatibility and equilibrium conditions are employed to relate the potential and normal flux on the interface between regions.[25]

The following integral equation, equivalent to (2), can be obtained from a reciprocity principle:[25]

$$c^i u^i+\int_\Gamma uq^*\,\mathrm{d}\Gamma=\int_\Gamma u^*\,q\,\mathrm{d}\Gamma, \tag{3}$$

where $c^i=\theta/2\pi$, θ is the internal angle at point i on the boundary Γ, u^* is the fundamental solution of Laplace's equation, $q=\partial u/\partial n$ and $q^*=\partial u^*/\partial n$.

The values of the potential u and its normal derivative q using cubic B-splines as interpolation functions are given at any point by

$$u(\omega)=E_0(\omega)a_{i-1}+E_1(\omega)a_i+E_2(\omega)a_{i+1}+E_3(\omega)a_{i+2}, \tag{4}$$

$$q(\omega)=E_0(\omega)b_{i-1}+E_1(\omega)b_i+E_2(\omega)b_{i+1}+E_3(\omega)b_{i+2}. \tag{5}$$

These expressions apply to the segment between nodes i and $i+1$; a_i are the coefficients (control points) of the potential representation and b_i the coefficients of the normal derivative; $E_k(\omega)$ are the blending functions for cubic B-splines given in Reference 1; and ω is a local parameter which varies between 0 and 1 within each segment. To represent corners, multiple knots are used and non-uniform blending functions must be applied as reported in Reference 2.

The present formulation employs isoparametric boundary elements for which the co-ordinates x and y of each point are also expressed by B-spline functions

$$x = E_0 X_{i-1} + E_1 X_i + E_2 X_{i+1} + E_3 X_{i+2}, \tag{6}$$

$$y = E_0 Y_{i-1} + E_1 Y_i + E_2 Y_{i+1} + E_3 Y_{i+2}, \tag{7}$$

where X and Y are the co-ordinates of the control points.

Application of the boundary integral equation (3) to all nodal points, incorporating approximations (4)–(7) within each boundary element, generates a system of algebraic equations which, in the present case, will have the form

$$\mathbf{Ha} = \mathbf{Gb}. \tag{8}$$

Evaluation of the coefficients of matrices **H** and **G** is discussed in detail by Cabral *et al.*[1,2]

Boundary conditions

The boundary conditions for each region can be of the following types:

specified potential:

$$u = \bar{u},$$

specified normal derivative:

$$q = \bar{q},$$

mixed condition (*semipervious layer*):

$$q = \frac{u - \bar{u}'}{c},$$

where $\bar{u}'$ is the specified potential above the semipervious layer and c its hydraulic resistance, i.e.

$$c = \frac{d}{K'},$$

in which d and K' are the thickness and the hydraulic conductivity of the semipervious layer, respectively.

Each of the above boundary conditions is related to the potential or flux, but generally the coefficients a_i and b_i are not known. So, the system (8) must be modified and the coefficients substituted by their relation with the potential and flux.

Writing expression (4) for all collocation points and collecting the result in matrix form gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & \cdots & \cdots & E_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{Bmatrix} \tag{9}$$

Each row of matrix **E** of nodal values of blending functions has three non-zero terms (four blending functions but one of them is null at each node), and all the other terms are zero.

Inverting equation (9) and substituting for vector **a** in (8), adopting a similar procedure for vector **b**, produces

$$\mathbf{HE}^{-1}\mathbf{u} = \mathbf{GE}^{-1}\mathbf{q}. \tag{10}$$

Phreatic surface. On the phreatic surface both values of u and q are unknown but the following mathematical relations can be stated:[26]

$$u=\eta(x, t), \tag{11}$$

$$\left[\frac{\partial u}{\partial t}\right]_x = -\frac{1}{\cos\beta_p}\frac{K}{\varepsilon}q+\frac{K}{\varepsilon}W \qquad \text{on } y=\eta, \tag{12}$$

where ε is the effective porosity, W is the vertical recharge intensity and β_p is defined as the angle between the phreatic surface profile and the x-axis.

Applying a finite difference approximation to represent the phreatic surface motion, one can state, following Reference 26,

$$u^{t+\Delta t}=u^t-\frac{K\,\Delta t\cos\delta}{\varepsilon\cos(\delta+\beta_p^t)}[\theta q^{t+\Delta t}+(1-\theta)q^t]+\frac{KW\Delta t}{\varepsilon}, \tag{13}$$

where Δt is the time interval and δ is the angle of displacement direction (Figure 3). This equation involves a linearization since the cosine of β_p is calculated at time t although the equation is written at time $t+\Delta t$. The use of small time steps is generally sufficient to produce good accuracy without iteration. Equation (13) provides an extra relation for the phreatic surface nodes which is applied to equation (10).

Interface. The next step in the formulation is to combine an equation of the type (10) for each region through the application of interface conditions. For each node along the interface between the freshwater and saltwater zones, the following conditions are used:

(1) Equilibrium of pressure, leading to

$$u_s=\frac{u_f}{\chi}+\frac{\chi-1}{\chi}y, \tag{14}$$

where $\chi=\rho_s/\rho_f$, ρ is the density and the subscripts f and s are related to the fresh and salt waters.

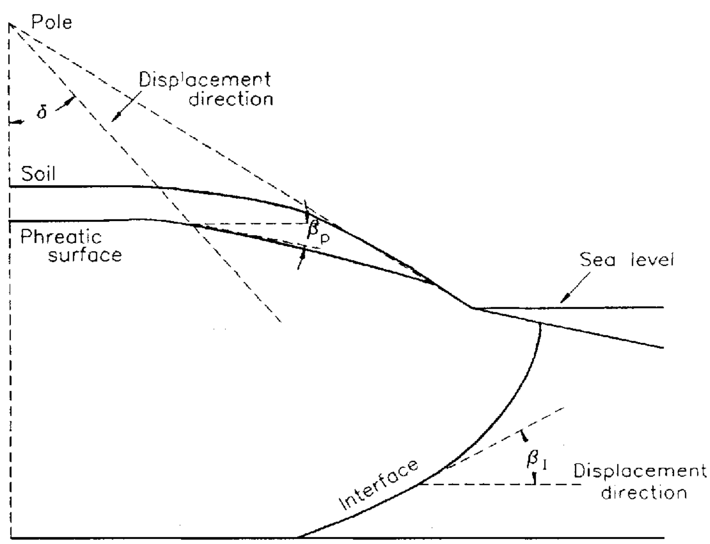

Figure 3. Interface and phreatic surface motions

(2) Compatibility of fluxes, which gives[26]

$$q_s = -\frac{\alpha}{\chi} q_f, \tag{15}$$

where $\alpha = \mu_s/\mu_f$ and μ is the dynamic viscosity.

Motion of interface and phreatic surface

The previous boundary conditions are all applied to system (10). At fixed boundaries either u or q, or a combination of both, is known at each point; at the phreatic surface expression (13) is applied to substitute $u^{t+\Delta t}$, and $q^{t+\Delta t}$ is determined as part of the solution; at the interface equations (14) and (15) are applied.

After $q^{t+\Delta t}$ at the phreatic surface has been determined from the solution of (10), the phreatic surface nodes are displaced according to the following relation:

$$\eta^{t+\Delta t} = \eta^t - \frac{K\,\Delta t \cos\delta}{\varepsilon\cos(\delta+\beta_p^t)}[\theta q^{t+\Delta t} + (1-\theta)q^t] + \frac{KW\Delta t}{\varepsilon}. \tag{16}$$

The normal derivatives at the interface are used to compute the interface motion.[26] Assuming $x = \lambda(y, t)$, the following equation can be written:

$$\frac{\partial\lambda}{\partial t} = -\frac{1}{\sin\beta_i}\frac{K}{\varepsilon} q_f, \tag{17}$$

in which β_i is the angle between the interface and the horizontal axis.

Applying a finite difference approximation to represent the interface motion, equation (17) becomes

$$\lambda^{t+\Delta t} = \lambda^t - \frac{\Delta t\,K}{\varepsilon\sin\beta_i^t}[\theta q_f^{t+\Delta t} + (1-\theta)q_f^t]. \tag{18}$$

Analogously to the phreatic surface case, the above equation also involves a linearization but the use of small time steps is generally sufficient to produce accurate results without iteration.

COMPUTATIONAL IMPLEMENTATION

Several important aspects regarding an efficient computer implementation of the B-spline BEM formulation were discussed in Reference 3, and also apply herein. Further points worth of notice, regarding this particular study, are discussed in what follows.

Remeshing

Due to the interface motion, the size of the freshwater and saltwater regions varies during the analysis. Thus, a node redistribution scheme ('remeshing') is needed after each time step. An efficient remeshing algorithm was developed which preserves the total number of elements and nodes of the initial discretization (although the number of elements and nodes in each region may change) and prevents elements located along the top and bottom boundaries becoming excessively large or excessively small, in comparison with their neighbours.

The basic ideas of the algorithm are as follows:

(1) The initial position of all nodes along the top and bottom boundaries is stored in memory.
(2) The interface toe displacement is calculated after each time step. The algorithm then finds the node in the *initial discretization* which is nearest to the displaced toe position, and moves this node.

(3) An analogous procedure is used for the interface tip.
(4) By always referring the interface toe and tip displacements to the initial discretization, elements and nodes are free to migrate from one region to the other. The algorithm keeps track of this movement, and assigns the boundary conditions at each node accordingly.

Tip motion

For the case in which the interface intersects a semipervious layer (Figure 1(a)), the normal derivative at the interface tip is computed similarly to the other interface nodes, and its motion follows the same procedure.

For other cases, e.g. when there is an outflow face (Figure 1(b)), the interface tip presents a singular behaviour. Bear[15] and Liu *et al.*[7] have shown that the interface should intersect the outflow face at a right angle. From the BEM point of view, the potential is known at the intersection point but there are two unknown fluxes; thus, an interpolated element[27] was used to overcome the problem. For calculating the location of the interface tip, an extrapolation was performed assuming a parabolic shape for the interface (following References 15 and 17). Alternatively, when using interpolated elements, the tip motion can be computed using the derivative normal to the interface.

Seepage face

For unconfined aquifers, it is important to note the tangency between the phreatic surface and the seepage face.[15] At the tangency point (Figure 2, point B), also called separation point, there is continuity of the slope and curvature, i.e. C^2 continuity.[28] Cubic B-spline elements can provide a good representation of these features using a single knot at the separation point. Then, at this point, the boundary condition $q=0$ was applied.

APPLICATIONS

Confined aquifer

The present BEM formulation was first applied to simulate Bear and Dagan's experiment[19] with a Hele–Shaw cell. Figure 4 shows a schematic representation of Bear and Dagan's experi-

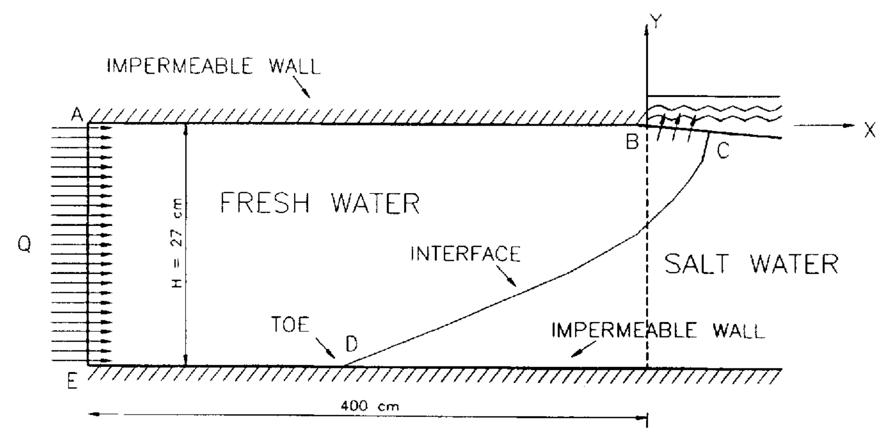

Figure 4. Recession of saltwater tongue by increased recharge

ment 3 in which the initial recharge rate of $Q=3{\cdot}9$ cm^2/s is increased to $Q=18{\cdot}8$ cm^2/s for $t>0$. Other characteristics of the problem are $\rho_f=1{\cdot}0$ g/cm^3, $\rho_s=1{\cdot}029$ g/cm^3, $K=69$ cm/s, $\varepsilon=1{\cdot}0$, $\mu=1{\cdot}2\times10^{-4}$ g/(cm s) and $H=27$ cm.

For the numerical analysis, an arbitrary position of the interface was initially assumed and a study undertaken to determine the initial steady-state configuration. Then, another analysis was performed, with the second recharge rate, to compute the transient behaviour of the saltwater tongue.

Figure 5 depicts the interface location at different times, compared to the experimental results of Bear and Dagan.[19] It can be seen that the numerical and experimental results predict the same pattern of interface movement although with different magnitudes. The present results, however, are in excellent agreement with other numerical solutions,[7,29] as shown in Figure 6. Moreover, as quoted by Liu *et al.*,[7] there is reason to believe that the experimental error may be significant. It is noted that Liu *et al.*[7] used 12 linear boundary elements along the interface compared with five B-spline elements employed here.

In Figure 7, a comparison is made between the two steady-state BEM solutions (for the two values of Q) and analytical results using Glover's equation as presented by van Dam.[17] The agreement with the analytical solution is very reasonable.

Leaky aquifer

The next application concerns a leaky aquifer on a coastal plain where the phreatic groundwater table (p) above the semiconfined aquifer is at the mean sea level (h_s) (see Reference 17, case d).

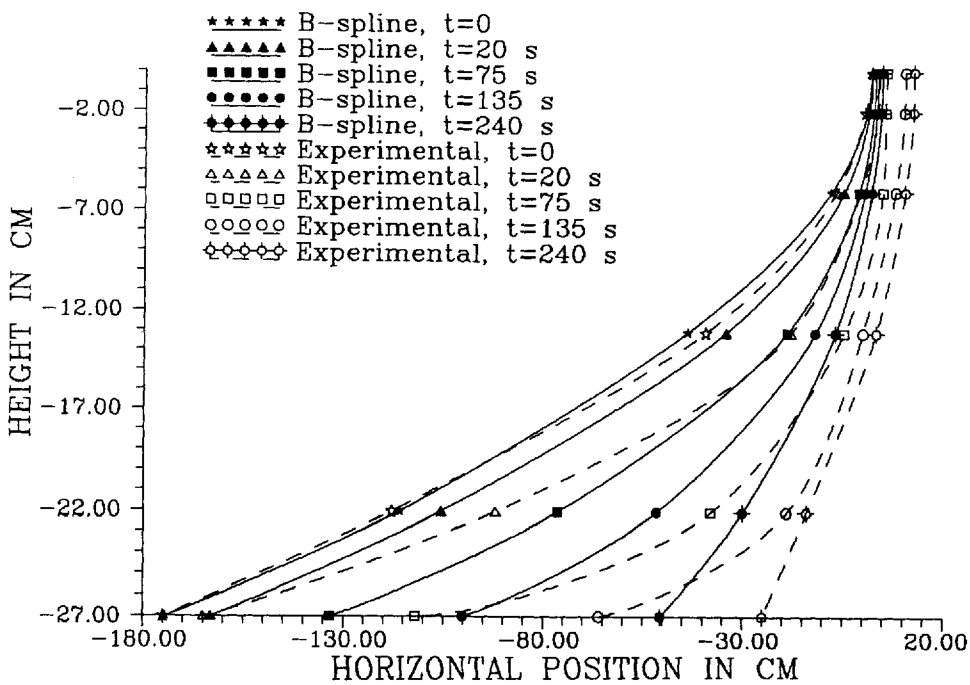

Figure 5. Transient behaviour of interface for confined aquifer (comparison with experimental data)

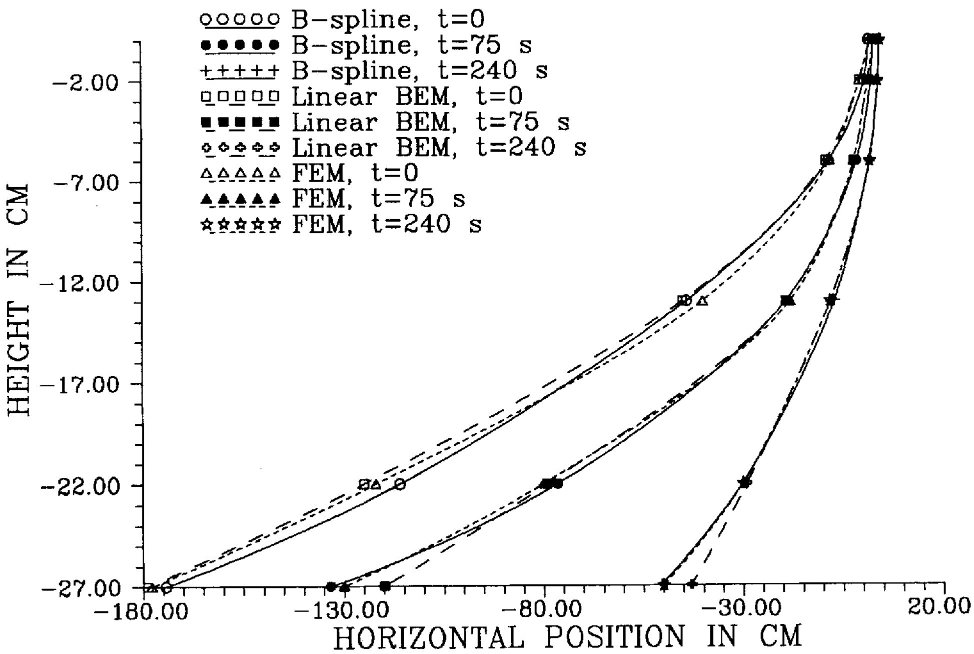

Figure 6. Transient behaviour of interface for confined aquifer (comparison with numerical data)

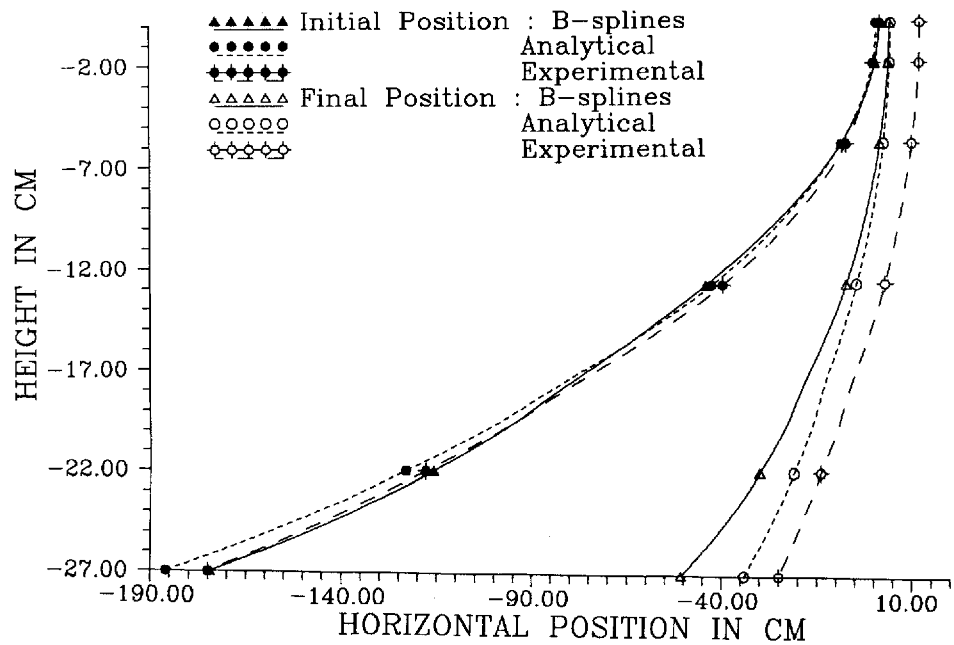

Figure 7. Steady-state positions of interface for confined aquifer

van Dam[17] presented an analytical solution to the one-dimensional steady flow under the Dupuit assumption which is as follows:

$$u=\chi\left[h_s+\frac{3E}{2}+\frac{(x-x_0)^2}{6Kc}\right],$$

$$Q=-\frac{\chi(x-x_0)^3}{18Kc^2}-\frac{\chi(x-x_0)E}{2c},$$

where Q is the groundwater flow rate per unit width (m^2/s), c is the hydraulic resistance of the semipervious layer and E is the equilibrium depth. For the case under study, $-E=h_s=p$ and Q can be related to the normal derivative q in the BEM formulation by $Q=q\times K\times$ aquifer thickness.

The results presented in what follows refer to the case depicted in Figure 1(a). The initial upstream recharge was assumed to be $Q=6{\cdot}78\times10^{-5}\,m^2/s$ ($q=0{\cdot}0339$); later, the recharge rate is increased to $Q=1{\cdot}7\times10^{-4}\,m^2/s$ ($q=0{\cdot}085$). Other physical characteristics of the problem are $\rho_f=1000\,kg/m^3$, $\rho_s=1029\,kg/m^3$, aquifer thickness $=200$ m, $K=10^{-5}$ m/s, $\varepsilon=0{\cdot}1$, $\mu=1{\cdot}2\times10^{-3}$ kg/(m s) and $c=10^8$ s.

For the numerical analysis, an arbitrary position of the interface was initially assumed and a study undertaken to determine the initial steady-state configuration. Then, another analysis was performed, with the second recharge rate, to compute the transient behaviour of the saltwater tongue.

Figure 8 depicts the interface location at different times, obtained with the B-spline formulation and with linear boundary elements.[12] The number of nodes along the interface was five in the B-spline analysis and seven with linear elements. It is interesting to note that, from $t=0$ to $t=42{\cdot}5$ days, a small change in the interface shape can be seen; at later times, the shape remains practically the same and only the interface position changes.

In Figure 9, a comparison is made between the initial and final steady-state positions (for the two values of Q), for the linear and B-spline boundary elements and the analytical solutions of van

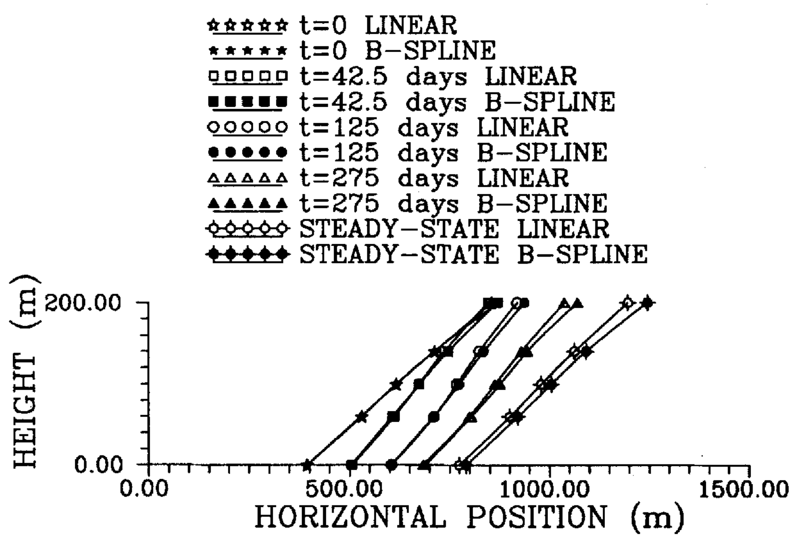

Figure 8. Transient behaviour of interface for leaky aquifer

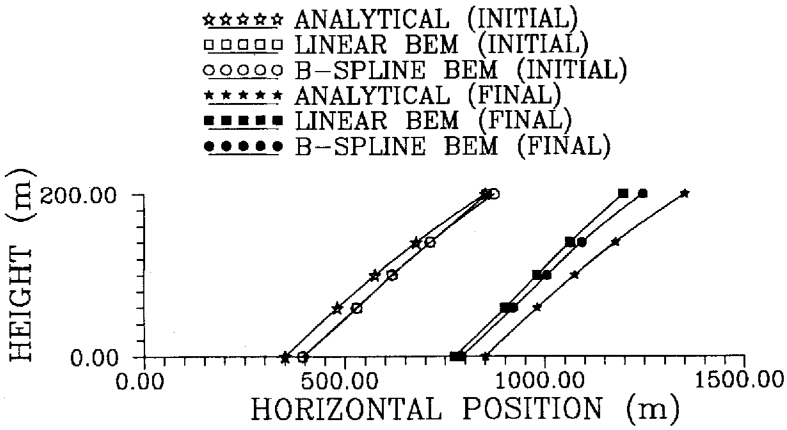

Figure 9. Steady-state positions of interface for leaky aquifer

Dam.[17] The agreement is very reasonable. It has been observed, however, that discrepancies between the numerical and analytical solutions appear as the value of Q increases, and become significant for large values of Q.[26]

Unconfined aquifer

For saltwater intrusion analysis in unconfined aquifers, Detournay and Strack[24] developed an interesting approximate analytical technique for the hodograph method. This technique allows an accurate computation of the phreatic surface, interface, seepage and outflow faces for simple configurations.

The present B-spline BEM formulation has been applied to one of their sample problems in which the slope of the seepage face (Figure 2, angle σ_1) is equal to 30°. The problem has been slightly modified by truncating the domain at the depth $y = -8$ m with the assumption of an impervious aquifer bottom, while Detournay and Strack's solution[24] is for an aquifer of infinite depth.

Other aquifer characteristics are: hydraulic conductivity $K = 40$ m/day, total freshwater outflow per unit width normal to the plane of flow $Q = 40\ \text{m}^2/\text{day}$, freshwater density $\rho_f = 1000\ \text{kg/m}^3$, saltwater density $\rho_s = 1400\ \text{kg/m}^3$ (similar to the Dead Sea), sea bottom slope $\sigma_2 = 15°$. Figure 10 shows that the seepage and outflow faces obtained by the two analyses are virtually the same, and good agreement between both solutions is also obtained for the phreatic surface and interface.

Finally, a transient analysis has been performed for an aquifer with the following characteristics:

hydraulic conductivity $K = 1{\cdot}16 \times 10^{-4}$ m/s
dynamic viscosity $\mu = 1{\cdot}2 \times 10^{-3}$ kg/(m s)
freshwater density $\rho_f = 1{\cdot}0 \times 10^3\ \text{kg/m}^3$
saltwater density $\rho_s = 1{\cdot}02 \times 10^3\ \text{kg/m}^3$
porosity $\varepsilon = 0{\cdot}05$
depth of aquifer bottom $D = 400$ m
initial precipitation recharge rate $W = 0{\cdot}002$ m/day
recharge rate in the second stage $W = 0$.

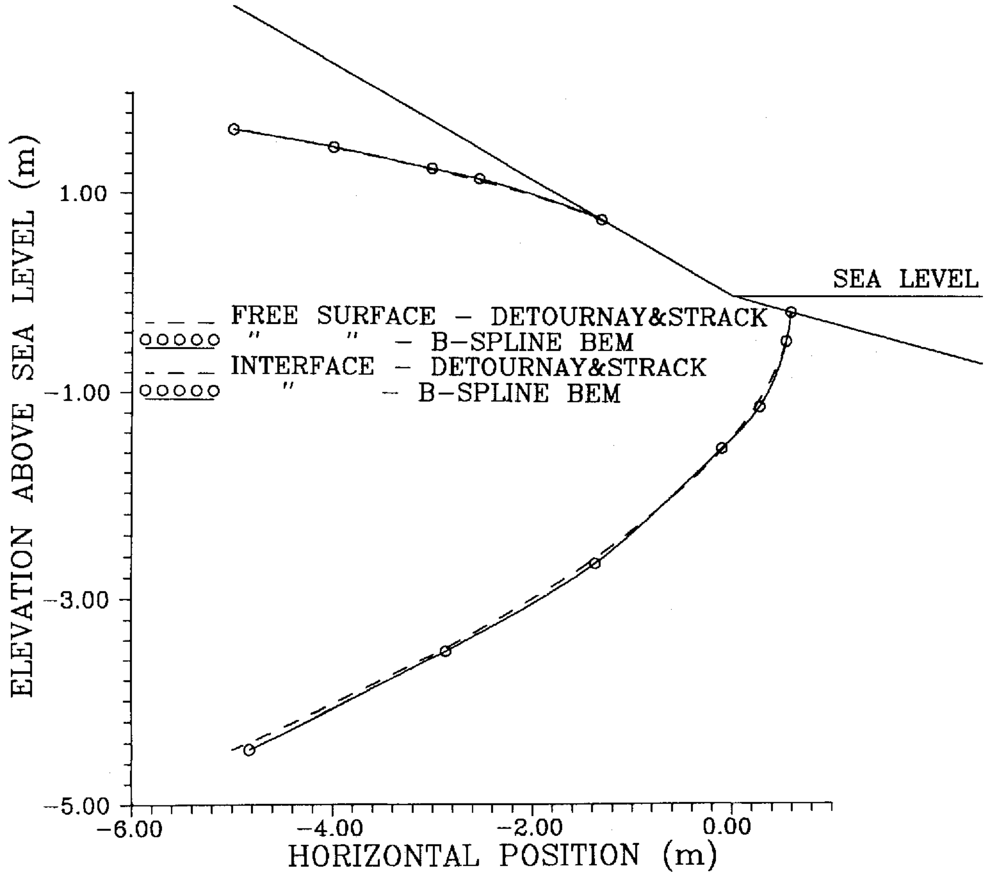

Figure 10. Steady-state configuration

The analysis was initiated with an arbitrary position of interface and phreatic surface, and an equilibrium configuration obtained for a precipitation recharge $W=0{\cdot}002$ m/day. This configuration was then adopted as initial position and the recharge rate modified to $W=0$, causing the interface and phreatic surface to start moving.

van der Veer[30] has developed the following analytical expression for the interface position:

$$H=\left[-\frac{\left(\frac{W}{K_f}x^2+2\frac{q^*}{K_f}x\right)}{\left(\frac{\chi-1}{\alpha'}+\frac{W}{K_f}\right)\left(\frac{\chi-1}{\alpha'}+1\right)}\right]^{1/2},$$

and for the phreatic surface:

$$h=\left\{-\left(\frac{W}{K_f}x^2+2\frac{q^*}{K_f}x\right)\left(\frac{\frac{\chi-1}{\alpha'}+\frac{W}{K_f}}{\frac{\chi-1}{\alpha'}+1}\right)-\left(\frac{q^*}{K_f}\right)^2\frac{\left[1-\left(\frac{\chi-1}{\alpha'}+\frac{W}{K_f}\right)\right]}{\left(1-\frac{W}{K_f}\right)\left(\frac{\chi-1}{\alpha'}+1\right)}\right\}^{1/2},$$

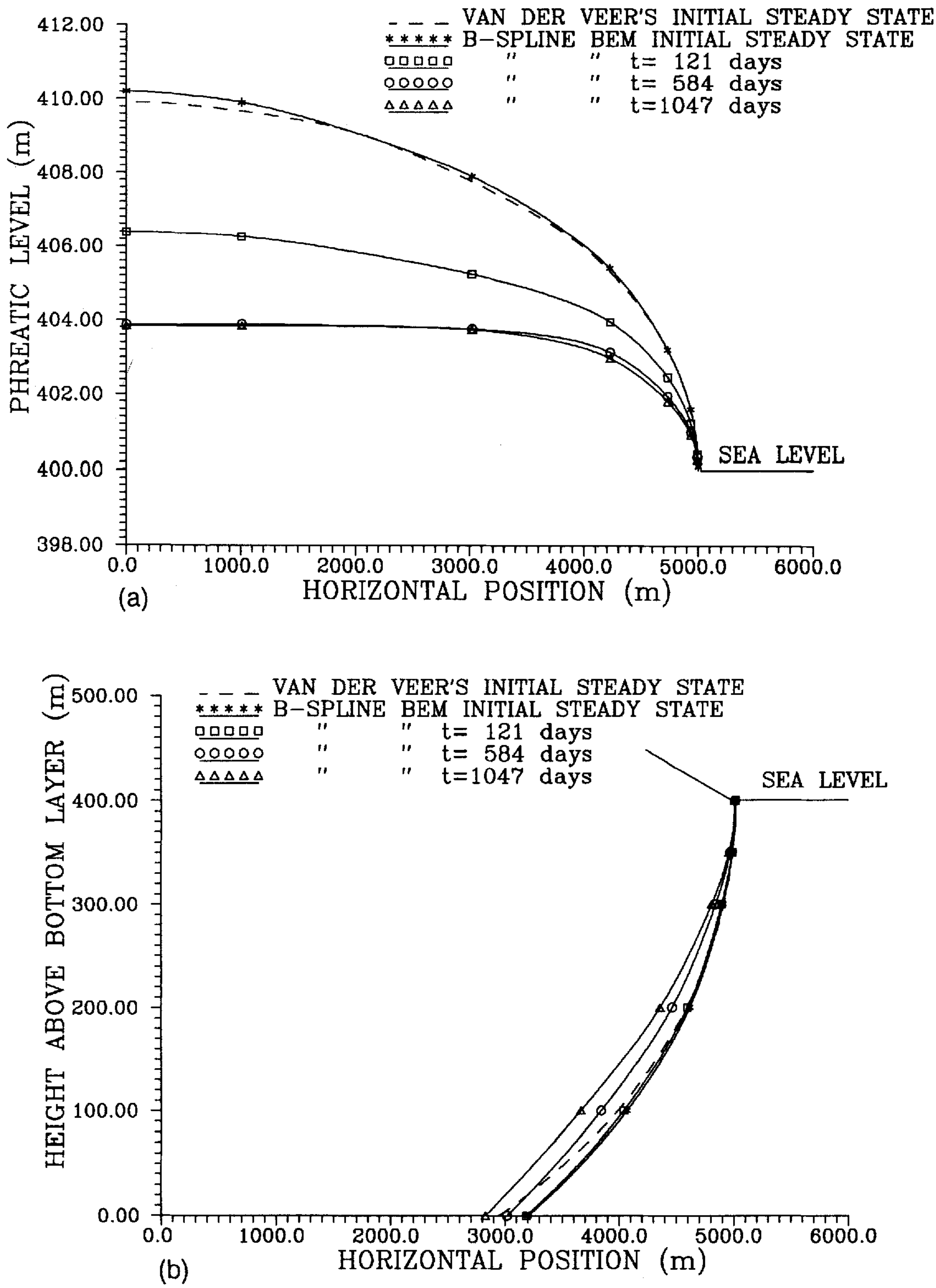

Figure 11. Saltwater intrusion in unconfined aquifer: (a) transient phreatic surface; (b) transient interface

where $q^* = W(L+L_e)+q_{\mathrm{upstream}}$, the subscript f refers to fresh water and the parameter α' refers to a relation between the motion of salt water and fresh water.

van der Veer's expressions[30] have been computed with parameter $\alpha' = 1{\cdot}0$. The results are plotted in Figure 11 together with the B-spline BEM equilibrium configuration and the phreatic surface and interface positions for different times.

Isaacs[18] and Isaacs and Hunt[31] derived an approximate analytical solution for the interface toe advance due to a replenishment reduction, in the form

$$x(t)=x(0)+\frac{0{\cdot}015}{x(0)/L+0{\cdot}12}\frac{KD}{\varepsilon L}t.$$

Figure 12 shows their results compared to the B-spline BEM in dimensionless form (position $x(t)/L$ versus time $tKD/\varepsilon/L^2$). It can be observed that the toe advance in the BEM formulation is smaller than that predicted by the above expression; however, Isaacs and Hunt[31] recognize that the above expression overestimates the toe advance.

CONCLUSIONS

For transient flow in porous media with a moving boundary (interface and/or phreatic surface), the normal derivative at points along the moving surface can be accurately computed using elements that provide derivative continuity. Although accurate results have been obtained with linear elements for some of the problems discussed in this paper, it is the author's experience that such elements can produce oscillations when modelling moving boundaries like the phreatic surface in unconfined aquifers, particularly when the seepage face is steep. No such problems have

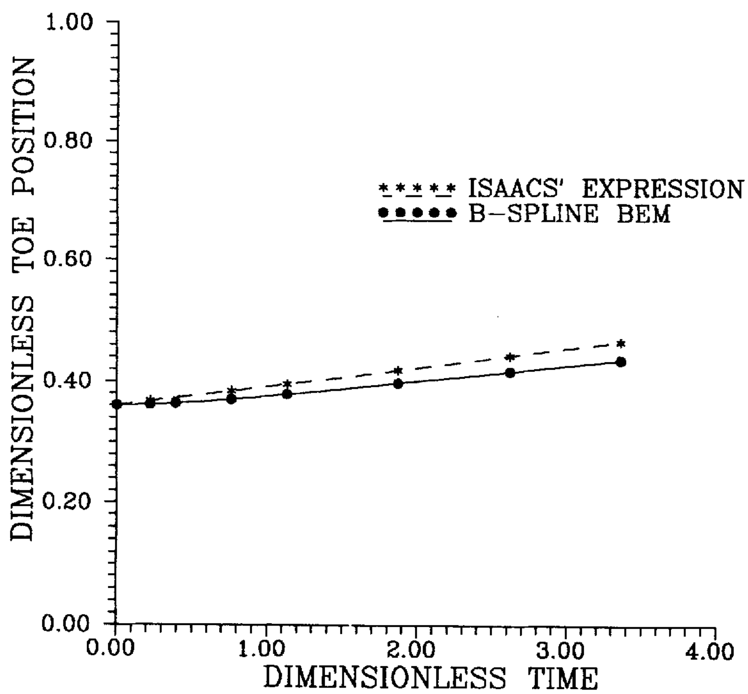

Figure 12. Interface toe motion

been observed with the present cubic B-spline formulation. In terms of computational efficiency, possibly the best solution would be a combination between linear elements on the fixed boundaries and B-spline elements along the moving boundaries.

The BEM formulation for saltwater intrusion has been extended to problems in unconfined aquifers which involve two interdependent moving boundaries. Special characteristics of saltwater intrusion analysis such as the determination of outflow and seepage faces have been accurately modelled using the present B-spline BEM formulation.

The model outlined in this paper can be used as a practical management tool to locate the saltwater wedge position by controlling the rate of upstream recharge.

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